Multi-unit Auctions

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- There are n bidders
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- For each bidder, total demand by other bidders (at price 0) exceeds k

Types of auctions

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- **Uniform-price auction**
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Discriminatory	Each winning bid pays bid amount
Uniform-price	Each winning bid pays highest losing bid
Vickrey	Winning bidders pays highest losing bids of other bidders

Auction Rules – example

$k = 3$, $n = 3$ and bids submitted are: Bidder 1: $(\mathbf{10}, \mathbf{8}, 6)$

Bidder 2: $(\mathbf{12}, 7, 0)$

Bidder 3: $(4, 0, 0)$

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Multi-unit auctions, single-unit demand

There are $n \geq k + 1$ bidders and each has demand for one unit only.

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Easy generalization of earlier results for single-object auction.

Multi-unit auctions, single-unit demand

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where Y_k is the kth highest of $\{X_2, X_3, \ldots, X_n\}$.

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Discriminatory auction: $b_d(x) = \int_0^x w(y, y) dL(y|x)$,

where $L(y|x) = \exp\left(-\int_y^x \frac{g(t|t)}{G(t|t)} dt \right)$ and

$g(y|x)$ is the density and $G(y|x)$ is the cdf of $Y_k = y$ given $X_1 = x$.

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With single-crossing, these auctions are efficient

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Bids submitted: (b_{i1}, b_{i2})

The auction selects either two bidders who get one unit each or one bidder who gets two units.

Vickrey auction

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The Vickrey auction is efficient. It is a dominant strategy for bidders to bid truthfully – that is, bid their valuations.

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Claim: It is a dominant strategy to bid truthfully: $(b_{i1}, b_{i2}) = (v_{i1}, v_{i2})$.

Proof: Let B_1 be the highest and B_2 the second-highest among bidders $2, 3, \ldots, n$ bids: $b_{21}, b_{22}, b_{31}, b_{32}, \ldots, b_{n1}, b_{n2}$.

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Efficiency in multi-unit auctions

Highest bids win. $k = 2$ objects, $n \geq 2$ bidders, $v_{i1} \geq v_{i2}$.

Let bid strategies be $b_{i1}(v_{i1}, v_{i2}), b_{i2}(v_{i1}, v_{i2})$.

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Conclusion: The Vickrey auction is efficient.

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Revenue equivalence

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An allocation rule of an auction is a function from bidder values to a probability distribution over the possible allocation of the units.

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An allocation rule of an auction is a function from bidder values to a probability distribution over the possible allocation of the units.

If two multi-unit auctions have the same allocation rule and the bidder with the lowest valuation has the same expected payoff in both auctions, then the expected revenue is the same in the two auctions.

Books

1. *Auction Theory* by Vijay Krishna, Academic Press.
2. *Putting Auction Theory to Work* by Paul Milgrom, Cambridge Univerity Press.
3. *Introduction to Auction Theory* by Flavio Menezes and Paulo Monteiro, Oxford University Press.
4. *Auctions: Theory and Practice* by Paul Klemperer, Princeton University Press.